# *Supplemental Material*

# Somigliana Coordinates: an elasticity-derived approach for cage deformation

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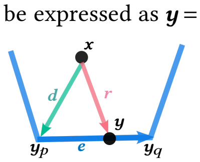
## A EVALUATING SOMIGLIANA COORDINATES

As explained in our main paper [Chen et al. 2023], Somigliana coordinates for each face $i$ and node $j$ are defined as

$$\begin{cases} T_i(\boldsymbol{x}) = \displaystyle\int_{\partial\Omega} \mathcal{T}(\boldsymbol{y},\boldsymbol{x})\phi_i(\boldsymbol{y})\, \mathrm{d}\sigma_{\boldsymbol{y}}, & \text{(1a)} \\ K_j(\boldsymbol{x}) = \displaystyle\int_{\partial\Omega} \mathcal{K}(\boldsymbol{y},\boldsymbol{x})\psi_j(\boldsymbol{y})\, \mathrm{d}\sigma_{\boldsymbol{y}}. & \text{(1b)} \end{cases}$$

Finding efficiently-computable closed-form expressions for these integrals is challenging. Instead, we apply quadratures to numerically evaluate them in practice. Here, we provide details on integrating these integrals for the 2D and 3D cases. Given a point $\boldsymbol{x} \in \Omega$, we compute both $T_i(\boldsymbol{x})$ and $K_j(\boldsymbol{x})$ by iterating over each element of the domain boundary.

***2D case.*** Given an edge $L_{pq} = (\boldsymbol{y}_p, \boldsymbol{y}_q)$, a point $\boldsymbol{y}$ on this edge can be expressed as $\boldsymbol{y} = \boldsymbol{y}_p + \alpha(\boldsymbol{y}_q - \boldsymbol{y}_p)$ for a barycentric coordinate $\alpha \in [0,1]$. We define $\boldsymbol{e} = \boldsymbol{y}_q - \boldsymbol{y}_p$, $\boldsymbol{d} = \boldsymbol{y}_p - \boldsymbol{x}$, and $\boldsymbol{r} = \alpha\boldsymbol{e} + \boldsymbol{d}$ (see inset), while $r = \|\boldsymbol{r}\|$ denotes the norm of $\boldsymbol{r}$. Since $\psi_j(\boldsymbol{y})$ are piecewise constant per edge, face coordinates are easily computed through quadrature as:




$$\begin{aligned} K_{L_{pq}}(\boldsymbol{x}) &= \int_{L_{pq}} \mathcal{K}(\boldsymbol{y},\boldsymbol{x})\, \mathrm{d}\sigma_{\boldsymbol{y}} = \|\boldsymbol{e}\| \int_0^1 \mathcal{K}(\boldsymbol{y}(\alpha),\boldsymbol{x})\, \mathrm{d}\alpha \\ &= \|\boldsymbol{e}\| \int_0^1 (a-b)\ln\left(\frac{1}{r(\alpha)}\right)\boldsymbol{I} + \frac{b}{r^2(\alpha)}\boldsymbol{r}(\alpha)\boldsymbol{r}^t(\alpha)\, \mathrm{d}\alpha \\ &\approx \|\boldsymbol{e}\| \sum_k w_k \left((a-b)\ln\left(\frac{1}{r(\alpha_k)}\right)\boldsymbol{I} + \frac{b}{r^2(\alpha_k)}\boldsymbol{r}(\alpha_k)\boldsymbol{r}^t(\alpha_k)\right), \end{aligned}$$

where $a = 1/\mu(2^{d-1}\pi)$ and $b = a/4(1-\nu)$, while $\{(w_k, \alpha_k)\}_k$ denotes the pairs of weights and (barycentric coordinates of) points of a Gauss-Legendre quadrature over the interval $[0,1]$.

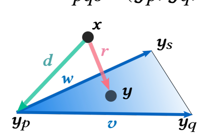
As for the vertex coordinates, integrating over the line segment $L_{pq}$ contributes $T_p^{L_{pq}}$ (resp. $T_q^{L_{pq}}$) to $T_p(\boldsymbol{x})$ (resp. $T_q(\boldsymbol{x})$) with:

$$\begin{aligned} T_p^{L_{pq}}(\boldsymbol{x}) &= \|\boldsymbol{e}\| \int_0^1 \mathcal{T}(\boldsymbol{y}(\alpha),\boldsymbol{x})(1-\alpha)\, \mathrm{d}\alpha \\ &= \|\boldsymbol{e}\| \int_0^1 \left\{ \frac{\mu(a-2b)}{r^2(\alpha)} \left[(\boldsymbol{n}^t\boldsymbol{r}(\alpha))\boldsymbol{I} + \boldsymbol{n}\boldsymbol{r}^t(\alpha) - \boldsymbol{r}(\alpha)\boldsymbol{n}^t\right] \right. \\ &\qquad \left. + \frac{4\mu b}{r^4(\alpha)}(\boldsymbol{n}^t\boldsymbol{r}(\alpha))\boldsymbol{r}(\alpha)\boldsymbol{r}^t(\alpha) \right\}(1-\alpha)\mathrm{d}\alpha. \end{aligned}$$

Similarly, we have $T_q^{L_{pq}}(\boldsymbol{x}) = \|\boldsymbol{e}\| \int_0^1 \mathcal{T}(\boldsymbol{y}(\alpha),\boldsymbol{x})\alpha\, \mathrm{d}\alpha$, and both of them are approximated using a Gauss-Legendre quadrature. We then assemble each $T_i(\boldsymbol{x})$ from its two neighboring edges $L_{pi}$ and $L_{iq}$ via:

$$T_i(\boldsymbol{x}) = T_i^{L_{pi}}(\boldsymbol{x}) + T_i^{L_{iq}}(\boldsymbol{x}).$$

***3D case.*** The 3D case proceeds similarly: given a triangular cage facet $\triangle_{pqs} = (\boldsymbol{y}_p, \boldsymbol{y}_q, \boldsymbol{y}_s)$, a point inside this triangle can be expressed using barycentric coordinates as $\boldsymbol{y} = \boldsymbol{y}_p + \alpha(\boldsymbol{y}_q - \boldsymbol{y}_p) + \beta(\boldsymbol{y}_s - \boldsymbol{y}_p)$, where $\alpha \in [0,1]$, and $\beta \in [0, 1-\alpha]$. Denote $\boldsymbol{v} = \boldsymbol{y}_q - \boldsymbol{y}_p$, $\boldsymbol{w} = \boldsymbol{y}_s - \boldsymbol{y}_p$, $\boldsymbol{d} = \boldsymbol{y}_p - \boldsymbol{x}$ (see inset), so that $\boldsymbol{r} = \boldsymbol{d} + \alpha\boldsymbol{v} + \beta\boldsymbol{w}$, with still $r = \|\boldsymbol{r}\|$. Then one has:




$$\begin{aligned} K_{\triangle_{pqs}}(\boldsymbol{x}) &= \int_{\triangle_{pqs}} \mathcal{K}(\boldsymbol{y},\boldsymbol{x})\mathrm{d}\sigma_{\boldsymbol{y}} \\ &= 2|\triangle_{pqs}| \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \mathcal{K}(\boldsymbol{y}(\alpha,\beta),\boldsymbol{x}) \\ &= 2|\triangle_{pqs}| \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \frac{a-b}{r(\alpha,\beta)}\boldsymbol{I} + \frac{b}{r^3(\alpha,\beta)}\boldsymbol{r}(\alpha,\beta)\boldsymbol{r}^t(\alpha,\beta) \\ &\approx 2|\triangle_{pqs}| \sum_k w_k \left(\frac{a-b}{r(\alpha_k,\beta_k)}\boldsymbol{I} + \frac{b}{r^3(\alpha_k,\beta_k)}\boldsymbol{r}(\alpha_k,\beta_k)\boldsymbol{r}^t(\alpha_k,\beta_k)\right), \end{aligned}$$

where $|\triangle_{pqs}|$ is the area of $\triangle_{pqs}$, while $\{w_k\}_k$ are the quadrature weights of their associated 2D quadrature points $\{(\alpha_k, \beta_k)\}_k$ on a canonical triangle in $\mathbb{R}^2$ with nodes located at $(0,0)$, $(0,1)$ and $(1,0)$. Taking into account both the cost and the generality of quadrature calculations to support high-order precision, we choose to subdivide each triangle into three quads to apply standard Gauss-Legendre quadratures — please refer to Sec. 5 of our paper for an illustration. Now, since the basis function $\phi_i(\boldsymbol{y})$ is a hat function, $T_i(\boldsymbol{x})$ is a sum of components coming from all its neighboring triangle facets, i.e., $T_i(\boldsymbol{x}) = \sum_{\triangle \in \mathcal{N}_i} T_i^{\triangle}(\boldsymbol{x})$. The contribution to $T_i(\boldsymbol{x})$ of a neighboring triangle $\triangle_{ipq}$, for instance, reads:

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$$\begin{aligned}T_i^{\triangle ipq} &= \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \mathcal{T}(\boldsymbol{y}(\alpha,\beta),\boldsymbol{x})(1-\alpha-\beta)\\ &= 2|\triangle_{ipq}|\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \left\{\frac{\mu(a-2b)}{r^3(\alpha,\beta)}\left[(\boldsymbol{n}^t\boldsymbol{r}(\alpha,\beta))\boldsymbol{I} + \boldsymbol{n}\boldsymbol{r}^t(\alpha,\beta)\right.\right.\\ &\quad \left.\left. - \boldsymbol{r}(\alpha,\beta)\boldsymbol{n}^t\right] + \frac{6\mu b}{r^5(\alpha,\beta)}(\boldsymbol{n}^t\boldsymbol{r}(\alpha,\beta))\boldsymbol{r}(\alpha,\beta)\boldsymbol{r}(\alpha,\beta)^t\right\}(1-\alpha-\beta).\end{aligned}$$

Similarly, we have

$$\begin{cases} T_j^{\triangle pjq} = \displaystyle\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \mathcal{T}(\boldsymbol{y}(\alpha,\beta),\boldsymbol{x})\alpha,\\ T_k^{\triangle pqk} = \displaystyle\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \mathcal{T}(\boldsymbol{y}(\alpha,\beta),\boldsymbol{x})\beta. \end{cases}$$

## B CONDITIONAL EQUIVALENCE TO CAUCHY-GREEN COORDINATES (2D GREEN COORDINATES)

In the special 2D case when $\nu = \pm\infty$ and $\gamma = 0$, the traction terms given by Eq. (14) from our paper is zero, so we only have the deformed boundary $\widetilde{\boldsymbol{y}}$ with traction kernel $\mathcal{T}_i(\boldsymbol{y},\boldsymbol{x})$ that contributes to the actual deformation $\widetilde{\boldsymbol{x}}$. Since $\lim_{\nu\to\pm\infty} b = 0$, traction kernel becomes $\mathcal{T}(\boldsymbol{r}) = \frac{1}{2\pi r^2}[(\boldsymbol{n}^t\boldsymbol{r})\boldsymbol{I} + \boldsymbol{n}\boldsymbol{r}^t - \boldsymbol{r}\boldsymbol{n}^t]$. As a result, the deformation computed by summing over all edges reduces to

$$\begin{aligned}\widetilde{\boldsymbol{x}}(\boldsymbol{x}) &= \frac{1}{2\pi}\sum_e \int_0^{L_e}\left(\frac{\boldsymbol{r}^t\boldsymbol{n}}{r^2}\boldsymbol{I} + \frac{1}{r^2}(\boldsymbol{n}\boldsymbol{r}^t - \boldsymbol{r}\boldsymbol{n}^t)\right)\widetilde{\boldsymbol{y}}\,\mathrm{d}\sigma_{\boldsymbol{y}}\\ &= \frac{1}{2\pi}\sum_e \int_0^{L_e}\frac{1}{r^2}\left\{\begin{pmatrix} r_1n_1+r_2n_2 & 0\\ 0 & r_1n_1+r_2n_2\end{pmatrix}\right.\\ &\qquad\qquad \left. + \begin{pmatrix} 0 & r_2n_1-r_1n_2\\ r_1n_2-r_2n_1 & 0\end{pmatrix}\right\}\widetilde{\boldsymbol{y}}\,\mathrm{d}\sigma_{\boldsymbol{y}}.\end{aligned} \tag{2}$$

For each edge, we can rewrite the above integral using complex numbers denoted with $\mathbb{hollow}$ letters, yielding

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{L_e}\frac{r_1n_1+r_2n_2+\mathbb{i}(r_1n_2-r_2n_1)}{\mathbb{r}^*\mathbb{r}}\widetilde{\mathbb{y}}\,\mathrm{d}\sigma_{\mathbb{y}}\\ =&\frac{1}{2\pi}\int_0^{L_e}\frac{(r_1-\mathbb{i}r_2)(n_1+\mathbb{i}n_2)}{\mathbb{r}^*\mathbb{r}}\widetilde{\mathbb{y}}\,\mathrm{d}\sigma_{\mathbb{y}}\\ =&\frac{1}{2\pi}\int_0^{L_e}\frac{\mathbb{r}^*\mathbb{n}}{\mathbb{r}^*\mathbb{r}}\widetilde{\mathbb{y}}\,\mathrm{d}\sigma_{\mathbb{y}} = \frac{1}{2\pi\mathbb{i}}\int_0^{L_e}\frac{\widetilde{\mathbb{y}}}{\mathbb{r}}\mathbb{i}\cdot\mathbb{n}\,\mathrm{d}\sigma_{\mathbb{y}}\\ =&\frac{1}{2\pi\mathbb{i}}\int_L\frac{\widetilde{\mathbb{y}}}{\mathbb{r}}\,\mathrm{d}\mathbb{y},\end{aligned} \tag{3}$$

where $\mathbb{r}^*$ denotes the conjugate of $\mathbb{r}$. This proves that when $\nu = \pm\infty$ and $\gamma = 0$, our 2D coordinates exactly reproduce the Cauchy-Green coordinates [Weber et al. 2009] derived from Cauchy's integral.

## REFERENCES

Jiong Chen, Fernando de Goes, and Mathieu Desbrun. 2023. Somigliana Coordinates: an elasticity-derived approach for cage deformation. In *Special Interest Group on Computer Graphics and Interactive Techniques Conference Conference Proceedings (SIGGRAPH '23 Conference Proceedings)*. 8 pages. https://doi.org/10.1145/3588432.3591519

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